SOME REMARKS ON LACUNARY INTERPOLATION IN THE ROOTS OF UNITY

BY

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ABSTRACT

The object of this note is to consider the problem of obtaining the explicit representations for polynomials of interpolation in the (0,2,3) case as explained in the introduction. We also show that Dini-Lipschitz condition suffices for the convergence problem, both in this and in the general result of Kis.

1. **Introduction.** In [2], we have considered the problem of finding explicit representation for polynomials $R_n(z)$ of degree $\leq 2n - 1$ which take, together with their third derivatives, certain preassigned values in the n^{th} roots of unity and the corresponding convergence problem. We call this the $(0,3)$ case and refer to [1] for references and notation. The object of this note is mainly to consider the $(0,2,3)$ case, but the method is capable of dealing with any lacunary case, e.g. $(0,r,r+k)$. Earlier, O. Kis [1] has already treated the $(0,2)$, $(0,1,3)$ and $(0,1,\cdots,r-2,r)$ cases in the roots of unity. However, to prove uniform convergence of his interpolatory polynomials in the two cases $(0,2)$ and $(0,1,3)$, he considers functions analytic inside the unit circle and continuous on the periphery, and the result in case $(0,2)$ differs from that in case $(0,1,3)$ by the requirement of a weaker condition on the modulus of continuity $\omega(\delta)$ on the unit circle. To be precise, he requires $\lim_{\delta\to 0}\omega(\delta)\log^2\delta = 0$. We shall show that this stronger requirement is not necessary and that the Dini-Lipschitz condition suffices.

In §2, we give the explicit form of the interpolatory polynomials in the $(0,2,3)$ case. §3 deals with the convergence problem. §4 is devoted to obtaining different forms for interpolatory polynomials in the $(0,1,3)$ case and to improvement of the theorem 4, of Kis [1].

Our method shows, although we do not do so explicitly, that the Dini-Lipschitz condition on $\omega(\delta)$ is sufficient for uniform convergence of interpolatory polynomials in any lacunary case – $(0,r,r+k)$ for example.

2. $(0,2,3)$ *case*. We shall prove the following:

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THEOREM 1. *If* $z_k = \exp(2\pi ik/n)$, $(k = 1,2,\cdots,n)$ *then the unique polynomial* $R_n(z)$ *of degree* $\leqq 3n-1$ *for which*

$$(1)\qquad R_n(z_\nu) = \alpha_\nu,\quad R''_n(z_\nu) = \beta_\nu,\quad R'''_n(z_\nu) = \gamma_\nu \qquad (\nu = 1,2,\cdots,n)$$

is given by

$$(2)\qquad R_n(z) = \sum_{k=1}^{n} \alpha_k A_{k,0}(z) + \sum_{k=1}^{n} \beta_k A_{k,2}(z) + \sum_{k=1}^{n} \gamma_k A_{k,3}(z)$$

where

$$(3)\qquad \begin{cases} A_{k,0}(z) = l_k(z) - (z^n - 1)[P_{k,0}(z) + z^n Q_{k,0}(z)] \\ A_{k,m}(z) = (z^n - 1)[P_{k,m}(z) + z^n Q_{k,m}(z)], \qquad m = 2,3 \end{cases}$$

and $P_{k,m}(z), Q_{k,m}(z), (m = 0,2,3)$ *are polynomials in* z *of degree* $\leqq n-1$ *given by*

$$(4)\qquad \begin{aligned} P_{k,m}(z) &= \frac{1}{6\beta}\int_0^1 t^{((2n-3)/2)-\beta}(1-t^{2\beta})F_{k,m}(tz)dt \\ Q_{k,m}(z) &= \frac{1}{6\beta}\int_0^1 t^{((2n-3)/2)-\beta}(1-t^{2\beta})G_{k,m}(tz)dt \end{aligned}$$

with $\beta = \frac{1}{6}\sqrt{3(34n^2-1)}$ *and formulas* (5) *and* (6) *giving the explicit forms for the polynomials* $F_{k,m}(u)$, $G_{k,m}(u)$:

$$(5)\qquad \begin{cases} F_{k,0}(u) = -\dfrac{1}{2n^2}[u^4 l_k^{(4)}(u) + 2(3n+2)u^3 l_k'''(u) + (n+1)(7n+2)u^2 l_k''(u)] \\ F_{k,2}(u) = \dfrac{z_k^2}{2n^2}[3u^2 l_k''(u) + 3(3n-1)u l_k'(u) + (n-1)(7n-2)\, l_k(u)] \\ F_{k,3}(u) = -\dfrac{z_k^3}{2n^2}\{2u l_k'(u) + (3n-1)\, l_k(u)\} \end{cases}$$

and

$$(6)\qquad \begin{aligned} G_{k,0}(u) &= \frac{1}{2n^2}[u^4 l_k^{(4)}(u) + 2(n+2)u^3 l_k'''(u) + (n+1)(n+2) l_k''(u)] \\ G_{k,2}(u) &= -\frac{z^2}{2n^2}[3u^2 l_k''(u) + 3(n-1)u l_k'(u) + (n-1)(n-2)\, l_k(u)] \\ G_{k,3}(u) &= \frac{z_k^3}{2n^2}\{2u l_k'(u) + (n-1)\, l_k(u)\}. \end{aligned}$$

Here as usual $l_k(u)$ denotes the fundamental polynomial of Lagrange interpolation and is given by

$$l_k(u) = (u^n - 1)\, z_k / n(u - z_k), \quad \text{where } z_k^n = 1.$$

Proof. (a) Since $A_{k,i}(z)$, $(i = 0,2,3)$ satisfy the conditions

$$(7)\qquad A_{k,0}(z_\nu) = \begin{cases} 1, & \nu = k \\ 0, & \nu \neq k \end{cases}; \qquad A_{k,0}^{(m)}(z_\nu) = o, \quad m = 2,3$$

$$(8)\qquad A_{k,2}^{(m)}(z_\nu) = 0, \quad m = 0,3; \quad A''_{k,2}(z_\nu) = \begin{cases} 1, & \nu = k \\ 0, & \nu \neq k \end{cases}$$

$$(9)\qquad A_{k,3}^{(m)}(z_\nu) = 0, \quad m = 0,2; \quad A'''_{k,3}(z_\nu) = \begin{cases} 1, & \nu = k \\ 0, & \nu \neq k \end{cases}$$

we set

$$A_{k,0}(z) = l_k(z) + (z^n - 1) r_{2n-1}(z)$$

where $r_{2n-1}(z)$ is a polynomial of degree $\leqq 2n - 1$. Then from (7) it follows that $r_{2\ -1}(z)$ satisfies the $2n$ conditions

$$(10)\qquad \begin{cases} 2nz_\nu r'_{2n-1}(z_\nu) + n(n-1)r_{2n-1}(z_\nu) + z_\nu^2 l''_k(z_\nu) = 0 \\ 3nz_\nu^2 r''_{2n-1}(z_\nu) + 3(n-1)nz_\nu r'_{2n-1}(z_\nu) + n(n-1)(n-2)r_{2n-1}(z_\nu) \\ \qquad = -z_\nu^3 l'''_k(z_\nu), \qquad (\nu = 1,2,\cdots,n) \end{cases}$$

where we have simplified the expressions, keeping in mind

$z_\nu^n = 1, (\nu = 1,2,\cdots,n)$. Setting $r_{2n-1}(z) = P(z) + z^n Q(z)$

where P, Q are polynomials of degree $\leqq n - 1$, we have

$$z_\nu^2 r''_{2n-1}(z_\nu) = z_\nu^2[P''(z_\nu) + Q''(z_\nu)] + 2nz_\nu Q'(z_\nu) + n(n-1)Q(z_\nu).$$

From (10) we have, on substituting these there, $2n$ conditions in Z_ν which give after simplification

$$(11)\qquad (2zD + n - 1)P(z) + (2zD + 3n - 1)Q(z) + \frac{z^2}{n} l''_k(z) = 0$$

and

$$(12)\qquad \{3z^2D^2 + 3(n-1)zD + (n-1)(n-2)\}P(z) + \\ + \{3z^2D^2 + 3(3n-1)zD + (n-1)(7n-2)\}Q(z) + \frac{z^3}{n} l'''_k(z) = 0$$

since each of the expressions on the left are polynomials of degree $\leqq n - 1$ which vanish in n points. Here $D \equiv d/dz$.

Differentiating (11), multiplying by $(3z/2)$ and subtracting from (11), we have

$$(13)\qquad \left\{\frac{3(n-3)}{2} zD + (n-1)(n-2)\right\} P(z) + \left\{\frac{3(3n-3)}{2} zD + (n-1)(7n-2)\right\} Q(z) \\ = \frac{z^3}{2n} l'''_k(z) + \frac{3z^2}{n} l''_k(z).$$

It is easy to verify that operators $\alpha zD+\beta$ and $\gamma zD+\delta$ are commutative when $\alpha,\beta,\gamma,\delta$ are constants. Then we have from (11) and (13) after some simplification, the following differential equation for $P(z)$:

$$[3z^2D^2+6nzD+(n-1)(2n-1)]P(z)=F_{k0,}(z)$$

where $F_{k,0}(z)$ is given in (5). Substituting $z=e^t$ and setting $\theta=(d/dt)$, we have

$$[3\theta^2+3(2n-1)\theta+(n-1)(2n-1)]y=F_{k,0}(e^t)$$

Since the roots of the auxiliary equation are $\alpha\pm\beta$ where $\alpha=-(2n-1)/2$ and $\beta=\frac{1}{6}\sqrt{3(4n^2-1)}$ we have after a change of variable

$$P(z)=\frac{1}{3\beta}\int_0^z\left(\frac{u}{z}\right)^{((2n-1)/2)}\sinh\left(\beta\log\frac{z}{u}\right)F_{k,0}(u)\frac{du}{u}$$

which on simplification gives (4) for $m=0$.

Similarly the differential equation for $Q(z)$ is found to be

$$[3\theta^2+3(2n-1)\theta+(n-1)(2n-1)]Q_{n-1}(e^t)=G_{k,0}(e^t)$$

where $G_{k,0}(u)$ is given by (6). This completes the proof of formula (4) for $m=0$.

(b) Set $A_{k,2}(z)=(z^n-1)s_{2n-1}(z)$ where $s_{2n-1}(z)=R(z)+z^nS(z)$, R, S being each polynomials of degree $\leqq n-1$. Then (8) leads as in (a) to the differential equations

$$(14)\qquad (2zD+n-1)R(z)+(2zD+3n-1)S(z)=\frac{z_k^2}{n}l_k(z)$$

$$(15)\qquad \left\{\frac{3(n-3)}{2}zD+(n-1)(n-2)\right\}R(z)+\left\{\frac{3(3n-3)}{2}zD+(n-1)(7n-2)\right\}S(z)$$
$$=-3z_k^2zl_k'(z)/2n$$

Then from (14) and (15) we get the differential equation for $R(z)$:

$$[3z^2D^2+6nzD+(n-1)(2n-1)]R(z)=F_{k,2}(z)$$

where $F_{k,2}(z)$ is given by (5). Similarly $S(z)$ satisfies a similar differential equation with the right side replaced by $G_{k,2}(z)$ as given by (6).

This completes the proof of (4) for $m=2$.

(c) Setting $A_{k,3}(z)=(z^n-1)t_{2n-1}(z)$ where $t_{2n-1}(z)=T(z)+z^n\mathscr{U}(z)$, $T,\mathscr{U}$ being polynomials of degree $\leqq n-1$, we get as in (a) and (b), the following:

$$(16)\qquad (2zD+n-1)T(z)+(2zD+3n-1)\mathscr{U}(z)=0$$

$$(17)\qquad \left\{\frac{3(n-3)}{2}zD+(n-1)(n-2)\right\}T(z)+\left\{\frac{3(3n-3)}{2}zD+(n-1)(7n-2)\right\}\mathscr{U}(z)$$
$$=\frac{1}{n}z_k^3l_k(z)$$

Here, as in (a) and (b), we get (4) for $m = 3$.

3. *Convergence Problem.* We shall now prove the following theorem:

THEOREM 2. *Let $f(z)$ be analytic in $|z| < 1$ and continuous for $|z| \leqq 1$. Let $\omega(\delta)$ be the modulus of continuity of $f(\exp ix)$, $(0 \leqq x \leqq 2\pi)$. If*

$$\lim_{\delta \to 0} \omega(\delta) \log \delta = 0 \tag{18}$$

and if

$$\beta_k = o(n^2/\log n), \quad \gamma_k = o(n^3/\log n), \quad k = 1, 2, \cdots, n \tag{19}$$

then the polynomial

$$R_n(z) = \sum_{k=1}^{n} f(z_k) A_{k,0}(z) + \sum_{k=1}^{n} \beta_k A_{k,2}(z) + \sum_{k=1}^{n} \gamma_k A_{k,3}(z)$$

where $A_{k,m}(z)(m = 0,2,3)$ are given by (3), converges uniformly to $f(z)$ in $|z| \leqq 1$.

We shall require the following:

LEMMA 1. *For $|z| \leqq 1$,*

$$\sum_{k=1}^{n} |A_{k,0}(z)| \leqq 169(3 + \log n) \tag{20}$$

$$\sum_{k=1}^{n} |A_{k,2}(z)| \leqq \frac{128}{n^2}(3 + \log n) \tag{21}$$

$$\sum_{k=1}^{n} |A_{k,3}(z)| \leqq \frac{40}{n^2}(3 + \log n). \tag{22}$$

We shall use the known estimate [1]

$$\sum_{k=1}^{n} |l_k(z)| \leqq 3 + \log n, \qquad |z| \leqq 1.$$

Also for $|u| \leqq 1$, this easily gives by a simple device and on applying Bernstein's inequality,

$$\sum_{k=1}^{n} |l_k^{(m)}(u)| \leqq n^m(3 + \log n), \quad m = 1, 2, 3, \cdots,$$

Now we have from (3), for $|z| \leqq 1$

$$\sum_{k=1}^{n} |A_{k,0}(z)| \leqq \sum_{k=1}^{n} |l_k(z)| + 2\sum_{k=1}^{n} |P_{k,0}(z)| + 2\sum_{k=1}^{n} |Q_{k,0}(z)|$$

$$\sum_{k=1}^{n} |A_{k,m}(z)| \leqq 2\sum_{k=1}^{n} |P_{k,m}(z)| + 2\sum_{k=1}^{n} |Q_{k,m}(z)|, \qquad m = 2, 3.$$

Also from (4), we have for $|z| \leqq 1$,

$$\sum_{k=1}^{n} |P_{k,0}(z)|$$

$$\leqq \frac{1}{6\beta} \int_0^1 t^{((2n-3)/2)-\beta} \max_{0 \leqq u \leqq 1} \sum_{k=1}^{n} |F_{k,0}(u)| dt.$$

and from (5),

$$\sum_{k=1}^{n} |F_{k,0}(u)|$$

$$\leqq \frac{1}{2n^2} \sum_{k=1}^{n} l_k^{(4)}(u)| + 2(3n+2) \sum_{k=1}^{n} |l_k'''(u)| + (n+1)(7n+2) \sum_{k=1}^{n} |l_k''(u)|$$

$$\leqq \frac{1}{2n^2}[n^4 + 2n^3(3n+2) + n^2(n+1)(7n+2)](3+\log n)$$

$$\leqq \frac{29n^2}{2}(3+\log n).$$

It is easy to check that $\frac{1}{6\beta} \int_0^1 t^{((2n-3/2)-\beta} dt \leqq \frac{4}{n^2}$ so that we have

$$\sum_{k=1}^{n} |P_{k,0}(z)| \leqq 58(3+\log n).$$

Similarly,

$$\sum_{k=1}^{n} |Q_{k,0}(z)| \leqq 26(3+\log n)$$

This enables us to obtain (20).

Now from (5) it is easy to see that for $|u| \leqq 1$,

$$\sum_{k=1}^{n} |F_{k,2}(u)| \leqq 11(3+\log n)$$

$$\sum_{k=1}^{n} |F_{k,3}(u)| \leqq \frac{3}{n}(3+\log n)$$

and from (6), it follows that for $|u| \leqq 1$,

$$\sum_{k=1}^{n} |G_{k,2}(u)| \leqq 5(3+\log n)$$

$$\sum_{k=1}^{n} |G_{k,3}(u)| \leqq \frac{2}{n}(3+\log n).$$

Then for $|z| \leqq 1$ we have

$$\sum_{k=1}^{n} |P_{k,2}(z)| \leqq \frac{44}{n^2}(3+\log n), \quad \sum_{k=1}^{n} |Q_{k,2}(z)| \leqq \frac{20}{n^2}(3+\log n)$$

$$\sum_{k=1}^{n} |P_{k,3}(z)| \leqq \frac{12}{n^3}(3+\log n), \quad \sum_{k=1}^{n} |Q_{k,3}(z)| \leqq \frac{8}{n^3}(3+\log n).$$

Since from (3)

$$\sum_{k=1}^{n} |A_{k,m}(z)| \leqq 2\sum_{k=1}^{n} |P_{k,m}(z)| + 2\sum_{k=1}^{n} |Q_{k,m}(z)|, \quad |z| \leqq 1, \quad m = 2,3$$

we at once get (21) and (22).

Proof of Theorem 2. The proof of Theorem 2 is now very easy. We consider $F_n(z)$, the Jackson means, and use the following estimates for them [1]:

$$|f(\exp ix) - F_n(\exp ix)| \leqq 6\omega(1/n) \tag{23}$$

$$|F_n^{(m)}(z)| \leqq 10(2n)^m \omega\left(\frac{1}{n}\right) \tag{24}$$

Then

$$\begin{aligned} f(z) - R_n(z) &= f(z) - F_n(z) + F_n(z) - R_n(z) \\ &= f(z) - F_n(z) + \sum_{k=1}^{n} (F_n(z_k) - f(z_k)) A_{k,0}(z) \\ &\quad + \sum_{k=1}^{n} (F_n(z_k) - \beta_k) A_{k,2}(z) \\ &\quad + \sum_{k=1}^{n} (F_n(z_k) - \gamma_k) A_{k,3}(z) \end{aligned}$$

so that using the estimates obtained above we have

$$\begin{aligned} |f(z) - R_n(z)| &\leqq 6\omega\left(\frac{1}{n}\right) + 6\omega\left(\frac{1}{n}\right) \times 169(3+\log n) \\ &+ \left\{40n^2\omega\left(\frac{1}{n}\right) + o\left(\frac{n^2}{\log n}\right)\right\}\frac{128}{n^2}(3+\log n) \\ &+ \left\{80n_6^3\omega\left(\frac{1}{n}\right) + o\left(\frac{n^3}{\log n}\right)\right\}\frac{40}{n^3}(3+\log n) = o(1) \end{aligned}$$

which proves the theorem.

4. (0,1,3) *case*. We now return to the case already treated by Kis, [1] for reasons explained in the introduction. Our purpose is to sketch a proof of the following theorem:

THEOREM 3. *If $f(z)$ is analytic in $|z| \leqq 1$, continuous in $|z| \leqq 1$, $\omega(\delta)$ being the modulus of continuity of $f(\exp ix)$, $o \leqq x \leqq 2\pi$ and if*

$$\lim_{\delta \to 0} \omega(\delta) \log \delta = 0$$

$$\alpha_k = o\left(\frac{n}{\log n}\right), \qquad \beta_k = o\left(\frac{n^3}{\log n}\right), \qquad k = 1,2,\cdots, n$$

then the polynomial

$$R_n(z) = \sum_{k=1}^{n} f(z_k) B_{k,0}(z) + \sum_{k=1}^{n} \alpha_k B_{k,1}(z) + \sum_{k=1}^{n} \beta_k B_{k,3}(z)$$

converges uniformly to $f(z)$ in $|z| \leqq 1$.

In order to prove this we shall have to find suitable forms for $B_{k,m}(z)$, $m = 0,1,3$, different from those obtained by O. Kis [1]. In view of the proof of Theorem 1 given above, we shall state without proof the following forms of the fundamental polynomials $B_{k,m}(z)$, $m = 0,1,3$:

$$(23) \qquad \begin{cases} B_{k,0}(z) = l_k(z) + (z^n - 1)[R_{k,0}(z) + z^n S_{k,0}(z)] \\ B_{k,m}(z) = (z^n - 1)\{R_{k,m}(z) + z^n S_{k,m}(z)\}, \qquad m = 1,3 \end{cases}$$

where

$$(24) \qquad \begin{cases} R_{k,0}(z) = -\dfrac{1}{n} z l_k'(z) - S_{k,0}(z) \\ R_{k,1}(z) = \dfrac{1}{n} z_k l_k(z) - S_{k,1}(z) \\ R_{k,3}(z) = -S_{k,3}(z) \end{cases}$$

and

$$(25) \qquad \begin{aligned} S_{k,0}(z) &= \frac{1}{2n^2} z^{1-n} \int_0^z t^{n-2} \left\{ \frac{2}{3} t^3 l_k'''(t) + (n+1) t^2 l_k''(t) + \frac{n^2-1}{3} t l_k'(t) \right\} dt \\ S_{k,1}(z) &= -\frac{z_k}{2n^2} z^{1-n} \int_0^z t^{n-2} \left\{ t^2 l_k''(t) + (n-1) t l_k'(t) + \right. \\ &\qquad \left. + \frac{(n-1)(n-2)}{3} l_k(t) \right\} dt \\ S_{k,3}(z) &= \frac{1}{6n^2} z_k^3 z^{1-n} \int_0^z t^{n-2} l_k(t)\, dt \end{aligned}$$

From these forms one easily formulates the following Lemma:

LEMMA 2.

$$\sum_{k=1}^{n} |B_{k,0}(z)| = 0(\log n)$$

$$\sum_{k=1}^{n} |B_{k,m}(z)| = 0(\log n/n^{m}), \qquad m = 1,3$$

The proof of Theorem 3 now follows exactly the same lines as in Theorem 2.

REFERENCES

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